Results on Sequences of Hermite-Padé, Intergal Approximant

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Abstract: The convergence of "horizontal" sequences of Hermite-Padé, Integral approximants is studied by both numerical and theoretical methods. For functions with their nearest sigularity of the algebraic type at $z = 1$ we complement previously proven convergence inside the unit circle by proving divergence outside the unit circle of sequences of the $[L/M; 1]$ type where M remains fixed and $L \to \infty$. We develop many new recursion relations for the integral and Hermite-Padé polynomials, and show how to obtain for them, and for previously known recursion relations, the coefficients in a computationally, relatively simple way.

Keywords: Hermite-Padé approximant, integral approximant, differential approximant, Padé approximant, differential equations, recursion relations.

1. Introduction and Summary

One of the main motivations for the study of integral approximants, and related Hermite-

Padé approximants is to provide efficient tools for the analysis of analytical functions which are known only by their associated Taylor series. The user of these techniques is often concerned with the extraction of the maximum amount of information from the limited number of series terms that che* has at cher disposal. Consequently some information is desirable with regards to efficiency, convergence properties, *etc.* We know [Stahl, 1987] that the diagonal, integral approximants converge in capacity, in regions determined by the structure of the series. An integral approximant is defined in terms of the integral polynomials. These in turn are defined by the linear algebraic equations (accuracy through order),

$$\sum_{j=0}^{k} Q_{j,\vec{r}}(z)\frac{d^j f(z)}{dz^j} - Q_{-1,\vec{r}}(z) = O\left(z^{s+1}\right). \tag{1.1}$$

In this paper we will not be concerned with the very real problems which occur when there is no unique solution to these defining equations. We use the notation Q_{-1} for the polynomial whose series coefficient is just unity. The degree of the polynomials is denoted by the vector subscript $\vec{r} = (l, m_0, m_1, \ldots, m_k)$. Here Q_{-1} is of degree l, and Q_j is of degree m_j for $0 \le j \le k$. In (1.1), the parameter,

$$s = l + \sum_{j=0}^{k}(m_j + 1). \tag{1.2}$$

The corresponding integral approximant, denoted by $[l/m_0; m_1; \ldots; m_k]$, is the solution of the differential equation

$$\sum_{j=0}^{k} Q_{j,\vec{r}}(z)\frac{d^j y(z)}{dz^j} - Q_{-1,\vec{r}}(z) = 0, \tag{1.3}$$

subject to the necessary boundary conditions to cause $y(z)$ to match $f(z)$ at the origin. The problem for Hermite-Padé approximants is similar. Here we have the defining equations for the

*Che is a third-person, singular, personal pronoun of indefinite gender.

Hermite-Padé polynomials,

$$\sum_{j=-1}^{k} f_j(z)Q_{j,\vec{r}}(z) = O\left(z^{s+1}\right), \tag{1.4}$$

where we have retained the range, $\{-1, \ldots k\}$ for consistency with (1.1). Again, the approximants are defined by the solution of,

$$\sum_{j=-1}^{k} y_j(z)Q_{j,\vec{r}}(z) = 0, \tag{1.5}$$

where the $y_j(z)$ are related to the $f_j(z)$ in a way that is appropriate to render (1.5) a solvable equation. An example would be $f_{-1}(z) = 1$, and $f_j(z) = [F(z)]^{j+1}$ for $0 \leq j \leq k$. The corresponding $y_j(z)$ would be $y_{-1}(z) = 1$ and $y_j(z) = [Y(z)]^{j+1}$ for $0 \leq j \leq k$ to yield an approximant $Y(z)$ to $F(z)$. This example is called algebraic approximants. For $k = 0$, both the algebraic and the integral approximants reduce to Padé approximants. Generally speaking the algebraic properties of the integral polynomials and the Hermite-Padé polynomials closely coincide, however the same need not be so of the approximants.

Since, in order to compute a diagonal integral approximant, quite a large number of terms of the defining series are required, the question naturally arises, could one get by with some sort of horizontal sequence which might require fewer terms to accomplish the same results. By a "diagonal" sequence, we mean that $l \approx m_0 \approx m_1 \approx \cdots \approx m_k \to \infty$, and by a "horizontal" sequence we mean that at least on index remains finite while the others tend to infinity. The first important question is one of convergence. That is, does the sequence being considered converge to the desired function, and over what region of the Riemann surface of the function? Likewise, when we suppose that there is convergence, then the question of the efficiency of the approximation arises. Put otherwise, how fast does it converge? In a previous paper, [2] we showed that the horizontal sequence $[L/M;1]$ with M fixed as $L \to \infty$, converged to the sought function in the unit circle and in certain lunes on other Riemann sheets with support in the unit circle. It is of interest to know if this result is sharp, and in what sense. Also it is of interest to know how the rate of convergence compares, using the same number of coefficients, to that of other horizontal and diagonal sequence. In the second section we give numerical examples using test functions to try to illuminate the answer to this question. From these examples, it appears that the sequence $[L/3;1]$ converges about as well as any of the other horizontal sequences, but that the horizontal sequences do not converge as well as the diagonal sequences when the same number of coefficients are used.

In the third section we find that some of the results of the second section are illusory. The sequence $[L/3;1]$ appears numerically to converge outside the unit circle. This result turns out to be an illusion caused by taking only 20 terms in the defining series. We prove in the third section that the sequences $[L/M;1]$ for the class of series considered must diverge outside the unit circle. There remain other important questions, such as the regions of convergence of horizontal sequences for which L and M both go to infinity for which we do not have any definite answers as yet.

Another question of interest to the person actually using integral approximants is that of their efficient computation. Here we contemplate the computation of the integral (or Hermite-Padé) polynomials, rather than the solution of the differential equations which is quite a different topic. Previously some recursion relations for this purpose had been known. The difficulty has been that the coefficients have been given in terms of determinants that were not especially easier to compute that the direct solution of the defining equations themselves. In the fourth section we show how to greatly expand the number of known identities between contiguous sets of integral polynomials and how to obtain the coefficients in a computationally relatively simple way.

2. Numerical Results

As a guide to what sort of results can be expected, we have computed numerically various ‘horizontal’ sequences of first-order, integral approximants to the test functions A-K of Hunter and Baker [7]. By a “horizontal” sequences $[l/m;n]$ we mean that n is fixed and finite while l or m or both tend to infinity. In particular we will choose $n=1$, so that we can compare with our previous results [2]. As a reminder to the reader of the results of that paper, let us be given a function $f(z)$ which satisfies the ordinary differential equation,

$$(1-z)f'(z)+G(z)f(z)=H(z), \tag{2.1}$$

where $H(z)$ and $G(z)$ are analytic in the disk $|z|\leq\rho$ for some $\rho>1$, and where $-G(1)$ is not an integer. It was proven that, if L is large enough, and the intergal polynomials for the $[L/M-1;1]$ are determined by,

$$[(1-z)+\alpha^{(L)}]f'(z)+g^{(L)}(z)f(z)-h^{(L)}(z)=O(z^{L+M+1}), \tag{2.2}$$

then,

$$\alpha^{(L)}=O(L^{-M-1}), \tag{2.3}$$

$$g^{(L)}(z)=\sum_{i=0}^{M-1}g_i^{(L)}(1-z)^i\rightarrow\sum_{i=0}^{M-1}G_i(1-z)^i,\quad \text{as } L\rightarrow\infty, \tag{2.4}$$

$$g_i^{(L)}-G_i=O(L^{-M+i}), \tag{2.5}$$

and

Table I. Test Functions $A-K$.

A	$(1-z)^{-1.5}+e^{-z}$
B	$(1-z)^{-1.5}(1+\frac{1}{2}z)^{1.5}+e^{-z}$
C	$(1-z)^{-1.5}(1-\frac{1}{2}z)^{1.5}+e^{-z}$
D	$(1-z)^{-1.5}+(1+\frac{1}{4}z^2)^{-1.25}+(1+\frac{15}{112}z-\frac{1}{4}z^2)^{-1.25}$
E	$(1-z)^{-1.5}(1+\frac{1}{2}z)^{1.5}+(1+\frac{1}{4}z^2)^{-1.25}+(1+\frac{15}{112}z-\frac{1}{4}z^2)^{-1.25}$
F	$(1-z)^{-1.5}(1-\frac{1}{2}z)^{1.5}+(1+\frac{1}{4}z^2)^{-1.25}+(1+\frac{15}{112}z-\frac{1}{4}z^2)^{-1.25}$
G	$(1-z)^{-1.5}+\{2(1-z)(2-z)^6/[(2-z)^7-z^7]\}^{1.25}$
H	$(1-z)^{-1.5}(1+\frac{1}{2}z)^{1.5}+\{2(1-z)(2-z)^6/[(2-z)^7-z^7]\}^{1.25}$
I	$(1-z)^{-1.5}(1-\frac{1}{2}z)^{1.5}+\{2(1-z)(2-z)^6/[(2-z)^7-z^7]\}^{1.25}$
J	$(1-z)^{-1.5}+(1+\frac{4}{5}z)^{-1.25}$
K	$(1-z)^{-1.5}+(1+\frac{4}{5}z)^{-1.25}+e^{-z}$

$$h^{(L)}(z)\to H(z)-\left[G(z)-\sum_{i=0}^{M-1}G_i(1-z)^i\right]f(z),\quad |z|<1, \tag{2.6}$$

as $L\to\infty$ where the G_i are the Taylor series expansion coefficients of $G(z)$ about $z=1$. In addition, it was proven that

$$\lim_{L\to\infty}[L/M;1]=f(z), \tag{2.7}$$

uniformly on compact subsets of $|z|<1$. It is of interest to know if this result is sharp, or if the region of convergence can be extended outside the unit circle, for this 'horizontal' sequence, or possible for some other 'horizontal' type sequence.

A further theorem which is relevant to our analysis was given by Baker, *et al.* [4]. It is, simplified to our present case, that if $G(z)$ of (2.1) is a polynomial of degree m_0, then, as $L\to\infty$ the $[L/m_0;1]$ converge to $f(z)$ uniformly on compact subsets of $|z|<\rho\setminus 1.0$. Barring unusual cancellations, we expect the same results to hold for the $[L/m;1]$ for $m\geq m_0$.

In table I, we repeat for the convenience of the reader the definitions of the test functions.

Table II. Differential Equation Coefficients for Test Functions $A-K$

	ρ	$G(z)$
A	∞	-1.5
B	2	$-4.5/(2+z)$
C	2	$-1.5/(2-z)$
D	1.75	-1.5
E	1.75	$-4.5/(2+z)$
F	1.75	$-1.5/(2-z)$
G	$1.109916\ldots$	-1.5
H	$1.109916\ldots$	$-4.5/(2+z)$
I	$1.109916\ldots$	$-1.5/(2-z)$
J	1.25	-1.5
K	1.25	-1.5

In order to analyze the behavior of the numerical results for the integral approximants to these test function, it is useful to display the coefficient $G(z)$ of eq. (2.1) for each of the functions. We do so in Table II, as well as the value of ρ, the smaller of the radii of convergence of the Maclaurin series of $G(z)$ and $H(z)$.

By the examination of Table II, we see that a horizontal sequence of integral approximants to the test functions, A, D, G, J, and K can be expected to converge in the disk $|z| < \rho$ except for the point 1.0 by the theorem of Baker *et al.* [4] quoted above. For the other test functions, we have only proven convergence in $|z| < 1$. Note is made that we could further expect for the other test functions that the $[L/0;2]$ will converge in $|z| < \rho$ by the just quoted theorem. Our point here is to investigate the behavior when convergence has not been established.

In all test cases above, we expect that the location and exponent will be found by the $[L/m;1]$

approximants as $L \to \infty$. In order to begin our comparison of the effectiveness of various sequences, we have tabulated in Table III, the number of decimal places of agreement with the exact value of the location of the singularity obtained by the use of N terms in the series expansion. In Table IV, we tabulate the same information for the exponent of the singularity, $\gamma = -G(1)$. By the number of decimal places we mean $\epsilon = -\log_{10}(|p - p_{\mathrm{exact}}|/|p_{\mathrm{exact}}|)$. Note is taken that ϵ can be negative if, for example, p is ten times p_{exact}, then the first digit is wrong, leading to a negative number of decimal places of agreement. The data for the "diagonal" case, $[L/M;N]$, where L, M, and N go to infinity together is taken from Hunter and Baker [7].

On reviewing the information in Tables III and IV, it appears that the near diagonal sequences are the best estimators of the critical properties. The other sequences were choosen to make approximately a fan in the $L-M$ approximant table. Among these other, non-diagonal sequences there does not appear to be much to choose from as far as effectiveness is concerned. As remarked above, convergence has been proven for the $[L/3;1]$ sequence, but the convergence is only like a power of L, rather than geometric. The behavior of the other non-diagonal sequences (for which we have, as yet, no rigorous results) appears to be similar. It is worth remarking that the approximants, as expected, work much more efficiently for test series A, where $\rho = \infty$, than they do when ρ is finite. The diagonal sequences have been proven to converge in capacity [10], but not pointwise as has been done for the $[L/M;1]$ sequences, M fixed, $L \to \infty$.

The analysis of Table V is somewhat more complex. By the theorem of Baker, *et al.* [4] mentioned above, we know there is convergence for series A, D, J, and K at regular points in the disk $|z| < \rho$ and so at the sample our points. For test case G, the same convergence holds, but our sample point was choosen outside that disk, though far from any singularity of the test

Table III. Comparison of Sequences – Singularity Location.

	N	diagonal	$[n/3;1]$	$[n/1+\frac{n}{3};1]$	$[n/n;1]$	$[\frac{n}{3}/n;1]$	$[3/n;1]$
A	10	3.6	2.8	2.5	2.8	2.3	2.8
	20	10.1	8.3	7.8	7.9	7.0	6.0
B	10	3.9	2.7	2.9	2.7	2.3	2.7
	20	8.9	5.4	4.8	5.7	5.7	5.8
C	10	4.1	3.2	1.9	3.2	2.2	3.2
	20	7.9	5.2	5.3	5.4	4.9	4.2
D	10	2.1	1.1	1.6	1.1	0.4	1.1
	20	5.3	2.7	2.9	2.2	0.5	2.2
E	10	2.7	1.1	−0.6	1.1	1.0	1.1
	20	4.8	3.2	3.2	3.0	2.6	2.0
F	10	1.8	1.5	0.1	1.5	0.8	1.5
	20	5.5	2.1	2.4	2.6	3.1	2.0
G	10	1.8	2.5	2.4	2.5	1.5	2.5
	20	4.4	1.5	1.6	1.7	2.6	2.7
H	10	2.5	2.5	2.5	2.5	1.7	2.5
	20	3.4	1.3	1.6	3.1	2.6	2.6
I	10	3.3	1.1	1.7	1.1	1.3	1.1
	20	3.3	1.6	1.6	1.2	1.2	3.2
J	10	3.7	0.6	−0.7	0.6	0.4	0.6
	20	8.7	1.9	0.6	0.3	0.2	1.3
K	10	3.4	−0.1	0.0	−0.1	−0.5	−0.1
	20	6.9	1.9	0.6	0.4	2.0	0.7

function. In the other cases B, C, E, F, H, and I, the convergence proven is that by Baker and Graves-Morris [2] and holds only for $|z| < 1$. Thus the sample points for all these cases lie

Table IV. Comparison of Sequences – Singularity Exponent.

	N	diagonal	$[n/3;1]$	$[n/1+\frac{n}{3};1]$	$[n/n;1]$	$[\frac{n}{3}/n;1]$	$[3/n;1]$
A	10	2.8	1.7	1.5	1.7	1.2	1.7
	20	8.4	6.6	6.2	6.2	5.3	4.3
B	10	3.1	1.6	1.9	1.6	1.2	1.6
	20	7.5	3.8	3.2	4.1	4.0	4.2
C	10	2.4	1.9	0.8	1.9	1.0	1.9
	20	6.2	3.6	3.7	3.7	3.3	2.7
D	10	1.7	−0.1	−0.1	−0.1	−1.4	−0.1
	20	3.6	1.4	1.6	0.6	−2.5	0.6
E	10	2.3	0.0	−3.4	0.0	−0.1	0.0
	20	3.3	2.0	2.0	1.4	1.2	0.4
F	10	1.3	0.3	−1.5	0.3	−0.5	0.3
	20	3.4	0.7	1.0	1.0	1.4	0.6
G	10	0.9	0.6	1.4	0.6	0.5	0.6
	20	3.9	0.5	0.5	0.2	1.8	1.0
H	10	1.9	1.6	1.1	1.6	0.8	1.6
	20	2.6	1.3	0.5	1.8	1.0	1.0
I	10	1.2	0.3	1.6	0.3	0.2	0.3
	20	2.1	0.5	0.5	−0.5	−0.5	2.9
J	10	3.0	−0.8	−3.1	−0.8	−1.4	−0.8
	20	7.4	0.6	−2.0	−2.6	−3.6	−0.3
K	10	1.7	−2.3	−1.9	−2.3	−4.2	−2.3
	20	5.5	0.6	−2.0	−3.0	0.6	−1.5

outside the region of proven convergence. From a causal glance it would appear that these cases (and G as well) are converging as well. In some of them, the convergence seems to be quite

Table V. Comparison of Sequences – Approximant Value. Values taken at $z = -1.5$, $A - H$, and at $z = -1.1$, J, K.

	N	$[n/3;1]$	$[n/1+\frac{n}{3};1]$	$[n/n;1]$	$[\frac{n}{3}/n;1]$	$[3/n;1]$
A	10	3.9	3.6	3.9	3.2	3.9
	20	11.9	9.9	11.7	9.7	7.7
B	10	3.6	3.6	3.6	3.1	3.6
	20	6.6	6.2	6.9	6.6	6.6
C	10	5.0	3.6	5.0	3.5	5.0
	20	7.5	7.4	8.7	7.2	5.4
D	10	0.7	0.5	0.7	0.6	0.7
	20	1.5	1.4	−9.2	*0.9	2.2
E	10	0.6	−31.3	0.6	0.9	0.6
	20	1.5	1.4	2.2	−5.1	−72.1
F	10	0.7	0.6	0.7	−0.8	0.7
	20	1.6	1.4	2.1	2.1	2.6
G	10	2.7	5.0	2.7	2.5	2.7
	20	5.2	5.0	*4.9	−93.1	1.5
H	10	3.3	3.5	3.3	2.4	3.3
	20	5.8	4.7	*4.4	−6.7	−17.9
I	10	2.7	4.9	2.7	2.3	2.7
	20	4.9	5.1	*5.5	*3.6	3.1
J	10	0.9	−11.1	0.9	0.7	0.9
	20	2.2	2.2	*1.6	*2.1	2.4
K	10	0.3	0.3	0.3	0.7	0.3
	20	2.3	2.3	*1.9	2.4	3.1

* Integration routine failure, previous value used.

Table VI. Approximant Value,
Test Series C.

n	$[n/3;1]$	$[n/n;1]$
1	3.1	2.2
2	4.2	4.0
3	5.0	5.0
4	5.2	5.4
5	5.7	5.6
6	5.7	7.3
7	5.6	7.8
8	6.3	8.7
9	6.9	
10	8.3	
11	7.7	
12	7.9	
13	7.5	

good. Note is taken that for some approximants the approximant values diverge to very large values, as reported here for test series D, E, G, H and J. There are also instances, presumably where the divergence is too strong for our integration routine (an adaptive step size Runge-Kutta method, [9], and it fails within the allowed minimum step size or the allowed maximum number of integration steps. These erratics (large values or integration routine failures) tend to be isolated in the sequences computed, except one can not be sure they are isolated when the last one or two entries in a sequence suffer in this way.

A more careful examination of the data raises a cautionary note. In table VI, we report the number of decimal places of agreement with the exact answer in more detail for test series C. The pattern of the increase in accuracy for the $[n/3;1]$ is not of the form of a uniform growth of accuracy which is to be expected inside a region of convergence. In fact the accuracy peaks at the $[10/3;1]$ approximant and declines subsequently. In contrast, the pattern of increase in

Table VII. Coefficients of the $g^{(L)}$ about $z = 1$,
Test Series C, Sequence $[n/n;1]$.

$[1/1;1]$	$[2/2;1]$	$[3/3;1]$	$[4/4;1]$	$[5/5;1]$	$[6/6;1]$	$[7/7;1]$	$[8/8;1]$
-1.755748	-1.982926	-1.525694	-1.505596	-1.500145	-1.499386	-1.451187	-1.500001
-1.473028	-2.682999	-0.252324	-0.093293	-0.005455	0.019562	-0.138642	-0.000054
	-0.327577	0.421910	0.408634	0.073721	-0.226986	0.132328	0.001434
		0.050524	0.047687	-0.093989	0.211376	-0.040840	-0.005972
			0.0012920	-0.020918	0.125310	-0.003876	0.005702
				-0.001373	0.020859	0.001793	-0.000068
					0.001209	0.000387	-0.000301
						0.000025	-0.000042
							-0.000002

accuracy, although a bit irregular, for the $[n/n;1]$ approximants is, relatively speaking, fairly steady. The results of the next section highlight the difficulties of sorting out the integral approximant convergence problems from numerical studies of the type that have served well for the Padé approximant case.

We have also looked to the convergence of the polynomials $g^{(L)}$. In the case of the $[n/3;1]$ sequence, the coefficients follow the results quoted at (2.3) - (2.5) above. That is the low order coefficients in the expansion in $(1 - z)$ converge fairly quickly, and the higher orders converge more slowly and to the expected results. In the case of the $[n/n;1]$ sequence, we see no particular pattern for the expansion about the origin for these coefficients. For the expansion about $z = 1$ there is a pattern of sorts. First the lowest order term apparently converges, leading to a convergence in the prediction of the properties at the singular point $z = 1$. Beyond that the magnitude of the coefficients drops off fairly rapidly. This behavior permits a wider circle of convergence (?!) than the straight expansion of $G(z)$ would do. For test series C the expansion coefficients of $G(z)$ about $z=1$ are, for reference, all equal to -1.5. We report in Table VII the values of the polynominal coefficients for this test series.

3. Divergence Theorem

The principle results of this section are to prove that for a certain class of functions, the convergence theorem of Baker and Graves-Morris [2] is sharp. Specifically the theorem is for the horizontal sequence $[L/M;1]$ of integral approximants with M fixed as L tends to infinity. The method is to prove divergence outside their proven radius of convergence.

In order to accomplish this result, we first need to use the Hermite form of the Lagrange interpolation formula. Consider the polynomial,

$$\begin{aligned}\psi(z) =& \frac{1}{2\pi i}\int_\Gamma \frac{t^{n+1}-z^{n+1}}{t-z}\frac{U(t)}{t^{n+1}}\,dt,\\ =&\frac{1}{2\pi i}\int_\Gamma \sum_{j=0}^{n} z^j t^{-1-j} U(t)\,dt,\\ =&\sum_{j=0}^{n}\frac{U^{(j)}(0)}{j!}z^j,\end{aligned}\tag{3.1}$$

by Cauchy's theorem, where Γ is any contour which encloses the origin. This form clearly generates the series expansion about the origin of $U(z)$ through order z^n.

THEOREM 3.1. *Assume that we are given a Maclaurin expansion to a function $f(z)$ which satisfies the equation (2.1) for $H(z)$ and $G(z)$ analytic in a disk $|z| \leq \rho$ for some $\rho > 1$. Further assume that $-G(1)$ is not an integer nor is $G(z)$ a polynomial. Then it follows that,*

$$\lim_{L\to\infty} |f(z) - [L-1/M-1;1]| = \infty, \tag{3.2}$$

for any given z in $1 < |z| < \rho$, and for any fixed $M \geq 0$.

PROOF: The case $M = 0$ is equivalent to forming the $[L-1/1]$ Padé approximant to $f'(z)$. It follows from Wilson's theorem [1] on the convergence for non-polar singularities that the Padé numerator tends to the power series about the origin for $(1-z)f'(z)$ and so diverges outside the unit circle. We will next treat the case $M \geq 1$. If we now apply (3.1) to (2.2), we obtain,

$$h^{(L)}(z) = \frac{1}{2\pi i\tau(z)} \int_\Gamma \frac{t^{L+M+1} - z^{L+M+1}}{t^{L+M+1}(t-z)} \{[(1-t) + \alpha^{(L)}]f'(t) + g^{(L)}(t)f(t)\}\tau(t)\, dt, \qquad (3.3)$$

where $\tau(z)$ is an arbitrary polynomial of degree $\leq M+1$. By the use of Cauchy's theorem on the first term of the integrand in (3.3), we can recast (3.3) into an error formula as,

$$[(1-z) + \alpha^{(L)}]f'(z) + g^{(L)}(z)f(z) - h^{(L)}(z) = \frac{z^{L+M+1}}{2\pi i\tau(z)} \int_\Gamma \frac{\{[(1-t) + \alpha^{(L)}]f'(t) + g^{(L)}(t)f(t)\}\tau(t)\, dt}{t^{L+M+1}(t-z)}. \qquad (3.4)$$

Let us now select the contour Γ to be the circle $|z| = \rho$ except for a small section near $z = \rho$ plus connecting line segments above and below $1 \leq z \leq \rho$ plus a small left semi-circle about $z = 1$ to close the contour. By means of the convergence described in Sec. 2 for $g^{(L)}(z)$ and $\alpha^{(L)}$ and the boundedness of $f(z)$ and $f'(z)$ as defined by (2.1) away from $z = 1$, we can bound the integrand of (3.4) on the circular portion of Γ by

$$\left| \int_{|z|=\rho\backslash\rho} \right| \leq \frac{z^{L+M+1}K_1\rho}{(\rho - |z|)\rho^{L+M+1}} \frac{\sup_{|t|=\rho} \tau(t)}{|\tau(z)|}, \qquad (3.5)$$

which tends geometrically to zero for $|z| < \rho$. Let us introduce the notation,

$$g_i^{(L)} - G_i = \hat{g}_i L^{i-M}, \quad \alpha^{(L)} = \hat{\alpha} L^{-M-1},$$

with $\alpha^{(L)}$,the g's, and the G's as given in Sec. 2. The next step is to consider the remaining portion of the contour Γ, *i.e.*, the part that wraps the real line $1 \leq z \leq \rho$. Next we can eliminate the $(1-t)f'(t)$ term from the integrand by use of (2.1). Since the form of the solution $f(z)$ of (2.1) is

$$f(z) = A(z)(1-z)^{-\gamma} + B(z), \tag{3.6}$$

with $A(z)$ and $B(z)$ both regular in $|z| \le \rho$, the contribution from $B(z)$ necessarily vanishes on the remaining portion of Γ. The exponent $\gamma = -G(1)$. In addition, as $H(z)$ is also regular in $|z| \le \rho$, the contribution from that term vanishes as well. If we now shrink the remainder of the contour on to the real axis and let $t = 1 + v$, we obtain,

$$\begin{aligned} &\frac{\sin(\pi\gamma) z^{L+M+1} L^{-M}}{\pi\tau(z)} \int_0^{\rho-1} \frac{\tau(1+v)\,dt}{(1+v)^{L+M+1}(1-z+v)} \\ &\times \left\{ L^{-1}\hat{\alpha}(\gamma v^{-\gamma-1}A + v^{-\gamma}A') + \sum_{j=0}^{M-1} \hat{g}_j (Lv)^j A v^{-\gamma} + \left[\sum_{j=0}^{M-1} G_j v^j - G(1+v)\right] A v^{-\gamma} \right\}. \end{aligned} \tag{3.7}$$

In order that the integrand be bounded, we choose the polynomial $\tau(z) = (1-z)^{\tilde{\gamma}+1}$, where $\tilde{\gamma}$ is defined as the least integer which is greater than or equal to γ. It is for this reason that we will subsequently require that $M \ge \gamma$. If instead, $M < \gamma$, then the integral in (3.7) diverges and the $v^{-\gamma-1}$ term diverges the most strongly. That result is sufficient to conclude the divergence outside the unit circle in the same manner as we argue below at (3.13) - (3.14). Next we notice that the structure of the integrand of (3.7) is such that it has terms which are functions of $V = Lv$, and those which are only functions of v. The convergence of the integral is provided by the $(1 + V/L)^{L+M+1}$ term. As the $\lim_{L\to\infty}(1 + V/L)^L = e^V$, we consider this term to be really a function of V and not just v. By standard arguments we may first ignore (to leading order in the $L \to \infty$ limit) the variation in v of those terms which depend on v in contrast to those which depend on V. Thus, to leading order in L as $L \to \infty$, we get, provide z is not close to $1 \leq z \leq \rho$,

$$\frac{\sin(\pi\gamma)z^{L+M+1}L^{-M}}{\pi(1-z)^{\tilde{\gamma}+1}}\int_0^\infty \frac{dv}{(1+V/L)^{L+M+1}}$$
$$\times\left\{L^{-1}\hat{\alpha}(\gamma v^{\tilde{\gamma}-\gamma}A+v^{\tilde{\gamma}-\gamma+1}A')+\sum_{j=0}^{M-1}\hat{g}_j(Lv)^j Av^{\tilde{\gamma}-\gamma+1}\right.$$
$$\left.+\left[\sum_{j=0}^{M-1}G_j v^j - G(1+v)\right]AL^M v^{\tilde{\gamma}-\gamma+1}\right\}. \quad (3.8)$$

To simplify subsequent presentation, we introduce the notation,

$$A(1+v)\left[G(1+v)-\sum_{j=0}^{M-1}G_j v^j\right]=\sum_{\lambda=0}^{\infty}a_\lambda v^\lambda\sum_{\mu=M}^{\infty}G_\mu v^\mu=\sum_{\nu=M}^{\infty}F_\nu v^\nu. \quad (3.9)$$

If we now use the β function integral,

$$\int_0^\infty \frac{x^{m-1}\,dx}{(1+x)^{m+n}}=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\asymp\frac{\Gamma(m)}{n^m}, \quad \text{for} \quad n\gg m, \quad (3.10)$$

then we can evaluate the integrals to obtain,

$$\frac{\sin(\pi\gamma)z^{L+M+1}L^{-M}}{\pi(1-z)^{\tilde{\gamma}+2}}\sum_{\nu=0}^{\infty}\left\{L^{-1}\hat{\alpha}(\gamma\Gamma(\tilde{\gamma}-\gamma+1+\nu)L^{\gamma-\tilde{\gamma}-1-\nu}a_\nu\right.$$
$$+\Gamma(\tilde{\gamma}-\gamma+2+\nu)L^{\gamma-\tilde{\gamma}-2-\nu}(\nu+1)a_{\nu+1}+\sum_{j=0}^{M-1}\hat{g}_j a_\nu\Gamma(\tilde{\gamma}-\gamma+2+\nu)L^{\gamma-\tilde{\gamma}-2-\nu}$$
$$\left.-F_\nu\Gamma(\tilde{\gamma}-\gamma+2+\nu)L^{\gamma-\tilde{\gamma}-2-\nu}\right\}. \quad (3.11)$$

We can compress the expression (3.11) as,

$$\frac{\sin(\pi\gamma)z^{L+M+1}}{\pi(1-z)^{\tilde{\gamma}+2}}O(L^{\tilde{\gamma}-\gamma-M-2}). \quad (3.12)$$

providing the leading order, $\nu=0$, does not cancel exactly. If we now compare the magnitude of (3.12) with that of (3.5), we see that in the limit of large L, for $|z|<\rho$, we may neglect (3.5) in comparison to (3.12). This comparison, of course, assumes that the leading order, $\nu = 0$ does not cancel exactly. Note that there are additional, higher-order terms in L^{-1} beyond those displayed in (3.11) arising from the v dependence of (3.7). Their occurrence will not however change the following discussion. Now as each each order in L^{-1} is a fixed, definite function of M, and the coefficients in (2.1), it is either zero or a definite non-zero number independent of L. They may not all vanish as that would imply that $f(z)$ is a solution of an ordinary differential equation of the form of (2.1) with $G(z)$ a polynomial, contrary to hypothesis. Thus there must exist a leading power in L^{-1}, whose exponent differs from that of (3.12) by a number which is a function of M at most of order $O(1)$ (with respect to L). Hence, for any $|z| > 1$ the right hand side of (3.4) diverges to infinity as $L \to \infty$. If we subtract the differential equation for the $[L-1/M-1;1]$ approximant,

$$[(1-z)+\alpha^{(L)}]y'(z)+g^{(L)}(z)y(z)-h^{(L)}(z)=0, \tag{3.13}$$

from (3.4), we obtain the differential equation for the error in the approximant, $d(z) = f(z) - y(z)$,

$$[(1-z)+\alpha^{(L)}]d'(z)+g^{(L)}(z)d(z)= \\ \frac{z^{L+M+1}}{2\pi i\tau(z)}\int_\Gamma \frac{\{[(1-t)+\alpha^{(L)}]f'(t)+g^{(L)}(t)f(t)\}\tau(t)\,dt}{t^{L+M+1}(t-z)}, \tag{3.14}$$

and so by the above results, the error diverges to ∞ for any $|z| > 1$, as was to be shown. ∎

We have not proved any results about the other sequences discussed in Sec. 2. However, we do remark that when the degree of the coefficient of $f(z)$ stays finite, then necessarily, the series for $h^{(L)}(z)$ in (2.2) must diverge outside the unit circle for the class of functions under discussion. We have proven this result above in considerable detail, but it was quite intuitive even before the proof. When both L and N go to infinity together with M fixed at unity in our current case, then existence of sequences of the form $[L/N;M]$ which converge in the compact subsets of $\{|z| < \rho \setminus 1\}$ is also quite clear. For example, the sequence where $g^{(L)}(z)$ is the first M terms of the Maclaurin expansion of $G(z)$ and $h^{(L)}(z)$ is the first L terms of the Maclaurin expansion of $H(z)$ will satisfy the above requirements. It is reasonable to hope that the corresponding integral approximants will converge as well in at least the same region. Furthermore if we rewrite (2.2) as

$$[(1-z)+\alpha^{(L)}]\left(\frac{f'(z)}{f(z)}\right) - h^{(L)}(z)\left(\frac{1}{f(z)}\right) + g^{(L)}(z) = O(z^{L+M+1}), \tag{3.15}$$

there would seem to be little intuitively to choose between sequences with $L > N$ and those with $N > L$. Consequently one would suppose that the best of the M fixed sequences for this class of functions would be the $L \approx N$ sequences. This idea is not contradicted by the evidence tabulated in Sec. 2.

4. Efficient Numerical Algorithms for Hermite-Padé (Integral) Polynomials

There are several known types of recursion relations [2,5,6] for the Hermite-Padé (integral) polynomials. To date, all the coefficients in these relations depend on the evaluation of determinants of the same general size as those required to solve for the integral polynomials directly. From the practical point of view, they are not very useful in their present form. In this section we show how they can be made more efficient in terms of the computational labor, and also point out how the same methods can be used to establish a number of additional recurrences between the integral polynomials. Generally in this section we will assume that all the integral polynomials considered are "essentially unique," *i.e.* unique except for an overall scale factor.

The cases where this property fails require substantial additional discussion.

The recurrence relations we shall discuss can be roughly grouped into two types. In the first type are those which do not depend on the series coefficients explicitly, put only through the coefficients of the "prior" integral approximants. This distinction is somewhat artificial, but we find it useful. Our first example can be thought of as belonging to either type and we use it to introduce the methods that we shall employ. The example is the recursion relation for $\lambda \neq \mu$, [2,6]

$$Q_{n,\vec{r}+\vec{\delta}_\mu+\vec{\delta}_\lambda}(z) D^{(0)}(\vec{r}) = D^{(0)}(\vec{r}+\vec{\delta}_\lambda) Q_{n,\vec{r}+\vec{\delta}_\mu}(z) - D^{(0)}(\vec{r}+\vec{\delta}_\mu) Q_{n,\vec{r}+\vec{\delta}_\lambda}(z), \tag{4.1}$$

where the subscript n tells which of the $n = -1, 0, \ldots, k$ integral polynomials is involved, the quantity $\vec{r}$ gives the position in the Hermite-Padé table, *e.g.* $\vec{r} = (L, m_0, m_1, \ldots, m_k)$ for example for the $[L/m_0; m_1; \ldots; m_k]$ integral approximant. Further the $\vec{\delta}_\mu$ are unit vectors in the direction μ in index space. The sign in (4.1) is correct for $\mu < \lambda$. The numbering refers to the order in which the corresponding series appear in the determinants $D^{(0)}(\vec{r})$. The key thing to notice about (4.1) is that the determinants are just numbers independent of z and n. In order for the relation (4.1) to hold, the first and second terms must combine to cancel the highest power of z for the case $n = \mu$, in order to agree with the required order in $Q_{\mu,\vec{r}+\vec{\delta}_\lambda}(z)$. Thus the ratio of the determinants is determined by the polynomial coefficients in the first two terms. Only the overall scale factor is undetermined, but the resulting polynomials can then be normalized by whatever rule ($Q_k(0) = 1.0$, for example) that one finds suitable for one's current purposes. In order to illustrate our other method, we will rederive (4.1) in a different way. First we introduce the notation,

$$\parallel \vec{r} \parallel \equiv \sum_{j=-1}^{k} (r_i + 1) - 1, \tag{4.2}$$

with the index $j = -1$ referring to the polynomial Q_{-1} whose series coefficient is just unity in the integral approximant case. Now consider the defining equations,

$$\sum_{j=0}^{k} Q_{j,\vec{r}+\vec{\delta}_\mu}(z) \frac{d^j f(z)}{dz^j} - Q_{-1,\vec{r}+\vec{\delta}_\mu}(z) = O\left(z^{\|\vec{r}\|+2}\right), \tag{4.3}$$

$$\sum_{j=0}^{k} Q_{j,\vec{r}+\vec{\delta}_\lambda}(z) \frac{d^j f(z)}{dz^j} - Q_{-1,\vec{r}+\vec{\delta}_\lambda}(z) = O\left(z^{\|\vec{r}\|+2}\right). \tag{4.4}$$

In (4.3) for example, the degree of Q_j is $\vec{\delta}_j \cdot (\vec{r} + \vec{\delta}_\mu)$. Similarly for (4.4). We can now determine the corresponding integral polynomials for the position $\vec{r} + \vec{\delta}_\mu + \vec{\delta}_\lambda$ in the approximant table by taking a linear combination of (4.3) and (4.4) as

$$\sum_{j=0}^{k} \left[aQ_{j,\vec{r}+\vec{\delta}_\mu}(z) + bQ_{j,\vec{r}+\vec{\delta}_\lambda}(z)\right] \frac{d^j f(z)}{dz^j} - aQ_{-1,\vec{r}+\vec{\delta}_\mu}(z) - bQ_{-1,\vec{r}+\vec{\delta}_\lambda}(z) = O\left(z^{\|\vec{r}\|+2}\right). \tag{4.5}$$

So far the equation (4.5) is one order short of determining the desired integral polynomials, however we can use the freedom of our choice of a and b to rectify this problem. The coefficient of $z^{\|\vec{r}\|+2}$ in (4.5) is selected to be,

$$\sum_{j=0}^{k} \sum_{i=0}^{r_j+\vec{\delta}_j\cdot(\vec{\delta}_\mu+\vec{\delta}_\lambda)} \left[aQ_{j,\vec{r}+\vec{\delta}_\mu,i} + bQ_{j,\vec{r}+\vec{\delta}_\lambda,i}\right] (\| \vec{r} \| + 3 - i)_j f_{\|\vec{r}\|+2+j-i} = 0, \tag{4.6}$$

where $(\mu)_j \equiv \mu(\mu+1)\cdots(\mu+j-1)$, and we have used the notation,

$$Q_{j,\vec{r}+\vec{\delta}_\mu}(z) = \sum_{i=0}^{r_j+\vec{\delta}_j\cdot\vec{\delta}_\mu} Q_{j,\vec{r}+\vec{\delta}_\mu,i} z^i. \tag{4.7}$$

Equation (4.6) determines the ratio of a to b. The overall scale factor determines normalization, and again we select that in a manner suitable for one's current purposes. The equation (4.1) uses the "determinant normalization." As (4.5) and (4.6) together are the defining equations for the integral polynomials of the type $\vec{r} + \vec{\delta}_\mu + \vec{\delta}_\lambda$, we can conclude that with an a and b which satisfy (4.6),

$$Q_{n,\vec{r}+\vec{\delta}_\mu+\vec{\delta}_\lambda}(z) = aQ_{n,\vec{r}+\vec{\delta}_\mu}(z) + bQ_{n,\vec{r}+\vec{\delta}_\lambda}(z). \tag{4.8}$$

It is to be noted that the first method described above, *i.e.* the one which does not involve solving an equation in which the defining series coefficients appear explicitly, works when the degree of contact with the defining series at the origin of the solved for integral polynomials, is less than or equal to that for the known integral polynomials. In the second method, the degree of contact is increased in the sought integral polynomials and here equations are involved that depend explicitly on the defining series coefficients.

One observes that, from the practical point of view, either method described above uses information currently available in the recurrence process and the computations are very much quicker than the evaluation of the determinants. With minor modifications, the same argument applies to the more general Hermite-Padé case. Specifically, the sum over j would also include the $j = -1$ terms, and the series coefficients are taken from the appropriate series rather than all computed from one basic series, $f(z) = \sum_{i=0}^{\infty} f_i z^i$, as in the integral approximant case, written out explicitly above.

Now we consider another identity that does not involve explicitly the defining series coefficients. Baker and Graves-Morris [3] showed that

$$Q_{n,\vec{r}+\vec{\delta}_\mu+\vec{\delta}_\lambda}(z)D^{(0)}(\vec{r}+\vec{\delta}_\theta) = Q_{n,\vec{r}+\vec{\delta}_\theta+\vec{\delta}_\mu}(z)D^{(0)}(\vec{r}+\vec{\delta}_\lambda) - Q_{n,\vec{r}+\vec{\delta}_\theta+\vec{\delta}_\lambda}(z)D^{(0)}(\vec{r}+\vec{\delta}_\mu), \quad (4.9)$$

In this identity, the degree of contact of every set of integral polynomials is the same. It is also symmetric between the θ, μ and λ directions. As it stands we can rewrite it as,

$$Q_{n,\vec{r}+\vec{\delta}_\mu+\vec{\delta}_\lambda}(z) = aQ_{n,\vec{r}+\vec{\delta}_\theta+\vec{\delta}_\mu}(z) + bQ_{n,\vec{r}+\vec{\delta}_\theta+\vec{\delta}_\lambda}(z), \quad (4.10)$$

and the ratio between a and b can be determined by cancelling the highest power of z in the Q_θ's on the right-hand side. Again the normalization can be fixed by what ever rule is convenient for present purposes. This identity is convenient to fill in a hyper-plane of constant degree of contact with the defining series from its boundary values. For example the first-order integral polynomials can be filled in from the Padé approximants alone to $f(z)$, $f'(z)$ and $f'(z)/f(z)$ by means of this identity.

Another interesting identity of this type which doesn't appear to have been found before is,

$$zQ_{n,\vec{r}}(z) = \sum_{j=-1}^{k} A_j Q_{n,\vec{r}+\vec{\delta}_j}(z). \tag{4.11}$$

The A_j are determined by equating the coefficients of the highest power of z on the left-hand side to that on the right-hand side for $n = j$. The integral polynomial defining equations for each term separately give zero for orders $z^0, \ldots, z^{\|\vec{r}\|+1}$. Hence, (4.11) will serve to give any one of the set of integral polynomials on the right-hand side (unnormalized) in terms of the remainder plus the one on the left-hand side. Note the interesting fact that as the left-hand side is divisible by z so too must the right-hand side be for each value of n. This fact is a consequence of the structure of the integral polynomial defining equations.

A number of the next class of identities have been given explicitly by Della Dora [5] for the case $k = 1$ in the Padé-Hermite context. They involve the embeddings of a linear chain of $k+3$ vertices (connected by $k+2$ bonds) on the index space describing the integral approximants. That is to say each vertex of this linear chain corresponds to an integral approximant and the bonds mean that location vectors for successive vertices differ by precisely unity in one and only one direction. These identities do not by any means exhaust the possibilities. In this class alone the number of types of possible identities, for which each successive vertex moves into a hyperplane of one order higher contact with the defining series, obeys the relationship $N(k,m) = N(k-1,m)m + N(k-1,m-1)$ where m is the dimension of the subspace spanned by the identity. For $k=0$ we have $N(0,1)=1,\ N(0,2)=1$. The sums over all possible m of $N(k,m)$ for $k=0,1,\ldots,6$ are 2, 5, 15, 52, 203, 877, and 3839. Manifestly, we will only discuss a minute fraction of these possibilities.

First we consider the "diagonal" type of identity. It is

$$Q_{n,\vec{r}+\sum_{i=-1}^{k}\vec{\delta}_i}(z) = azQ_{n,\vec{r}}(z) + A_{-1}Q_{n,\vec{r}+\vec{\delta}_{-1}}(z) + A_0 Q_{n,\vec{r}+\vec{\delta}_{-1}+\vec{\delta}_0}(z) + \cdots + A_{k-1}Q_{n,\vec{r}+\sum_{i=-1}^{k-1}\vec{\delta}_i}(z). \quad (4.12)$$

Any permutation of the order of adding directions is equally valid. In particular, the cyclic permutations will be required to advance, step by step along the diagonal in the integral approximant table. The same remarks apply to Hermite-Padé approximants. The degree of contact on the right-hand side of (4.12) is equal to or greater than that for the second term on the right-hand side, because the first term has a factor of z. In like manner to (4.6) we impose the requirements, that the coefficients of $z^{\|\vec{r}\|+2},\ldots,z^{\|\vec{r}\|+2+k}$ on the right-hand side vanish in the integral approximant defining equations. As the coefficients of orders $z^0,\ldots,z^{\|\vec{r}\|+1}$ vanish by construction, these additional equations insure that the integral polynomial defining equations are satisfied for the integral polynomials on the left-hand side of (4.12). There are also the correct number of equations to determine, except for the over all normalization, the multipliers $a, A_{-1}, A_0, \ldots, A_{k-1}$. There results a set of $k+1$ linear equations to be solved for these multipliers, which is computationally much simpler than the direct solution of the complete set of integral polynomial defining equations. This mode of derivation also establishes the existence of these identities, without the prior knowledge which we have for $k=0,1$ that they exist, because we have shown that the derived polynomials directly satisfy the integral polynomial defining equations. We feel that this is a powerful method to find identities among the set of integral polynomials. For small values of k, like 0 or 1 for instance, it is quite practical to solve the equations directly and give explicit expressions for the multipliers.

There are other "space types" in this class of identities. That is to say, other embeddings of the $k+2$ step linear chain on the index-space lattice that lead to interesting identities. For example there is the case where a single index alone increases. Here the result is,

$$Q_{n,\vec{r}+(k+2)\vec{\delta}_\mu}(z) = azQ_{n,\vec{r}}(z) + \sum_{j=1}^{k+1}(A_j + B_j z)Q_{n,\vec{r}+j\vec{\delta}_\mu}(z). \tag{4.13}$$

In this case there are $2k+3$ constants on the right-hand side to be determined. Again, the polynomial defining equations of orders $z^0, \ldots, z^{\|\vec{r}\|+1}$ are satisfied by construction. We must impose the additional $k+1$ accuracy-through-order equations, for the orders, $z^{\|\vec{r}\|+2}, \ldots, z^{\|\vec{r}\|+2+k}$. In addition in this case in order to keep the left-hand side of (4.13) at its stated order, we require that the highest power of z cancel between the terms on the right-hand side for the cases $n = -1, 0, \ldots, k$ except for $n = \mu$. This condition leads to $k+1$ simultaneous linear equations between the B_j and a, which are thereby determined, except for over all normalization. The A_j can then be determined from the B_j and the accuracy-through-order equations mentioned above. This case therefore requires the solution of two sets or $k+1$ equations rather than one as in the previous case. It is never the less still computationally much simpler than the direct solution of the complete set of integral polynomial defining equations.

Identity (4.13) is an example of what we called "reduced dimension" identities. It has the property that the all the integral polynomials lie in a subspace of the index space that is of less than full dimension. In the case of (4.13) it is a one-dimensional subspace. Another example is the identity,

$$Q_{n,\vec{r}+\vec{\delta}_\lambda+(k+1)\vec{\delta}_\mu}(z) = azQ_{n,\vec{r}}(z) + A_{k+1}Q_{n,\vec{r}+(k+1)\vec{\delta}_\mu}(z) + \sum_{j=1}^{k}(A_j + B_j z)Q_{n,\vec{r}+j\vec{\delta}_\mu}(z). \quad (4.14)$$

The difference here from (4.13) is that $B_{k+1} \equiv 0$ and the number of equations needed to cancel extra powers of z on the right-hand side is reduce by one (the case $n = \lambda$). Other than that the discussion is identical. The structure of (4.14) will be seen to be intermediate between that of (4.12) and (4.13). We give one further such identity. It is again in a two dimensional subspace. It is the analogue of the "stair-step" identities of Padé approximant theory. First let us define

$$\vec{r}_m = \begin{cases} \vec{r} + \frac{1}{2}m(\vec{\delta}_\mu + \vec{\delta}_\lambda), & m \text{ even}, \\ \vec{r} + \vec{\delta}_\mu + \frac{1}{2}(m-1)(\vec{\delta}_\mu + \vec{\delta}_\lambda), & m \text{ odd}. \end{cases} \quad (4.15)$$

Then the identity is

$$Q_{n,\vec{r}_{k+2}}(z) = azQ_{n,\vec{r}}(z) + A_{k+1}Q_{n,\vec{r}_{k+1}}(z) + \sum_{j=1}^{k}(A_j + B_j z)Q_{n,\vec{r}_j}(z). \quad (4.16)$$

Here the structure is the same as in (4.14), and the method of derivation of the coefficients is also the same.

There is one item to be noted at this point. That is, that in the above derivations, it is assumed that all the polynomials are non- zero. For instance, in (4.14) if $\vec{\delta}_\lambda \cdot \vec{r} = -1$ so that the integral polynomials, $Q_\lambda \equiv 0$ on the right-hand side of the equation, then manifestly, we can not produce a non-zero one on the left-hand side of the equation. We see here the "reduction effect." Namely, the identity in this case involves only integral polynomials in a smaller subspace, so that the right-hand side vanishes identically for such a case, for all n. This prevents for example, one from using the Padé polynomials for the $[N/N]$ to compute in only a few steps the integral polynomials for the $[N/N;0]$ or the $[N/N;1]$.

The corresponding identity to (4.11) "in the other direction" is known in the determinantal form [2,6]. It is not a "linear chain identity," but one of the plethora of other possibilities. Here it is not necessary to us to involve $k+3$ different sets of integral polynomials. Thus we can get a family of identities. They are,

$$Q_{n,\vec{r}}(z) = \sum_{j=1}^{m} A_j Q_{n,\vec{r}-\sum_{i=1}^{m}\vec{\delta}_{\mu_i}+\vec{\delta}_{\mu_j}}(z). \tag{4.17}$$

In this case the A_j are determined, except for the overall normalization, by equating the coefficients in the integral polynomial determining equations to zero for the orders, $z^{\|\vec{r}\|-m+1}, \ldots, z^{\|\vec{r}\|}$. Identity (4.1) is the special case, $m=2$, of (4.17).

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